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Function Estimation in Predicting Temperature-Dependent Thermal Conductivity Without Internal Measurements

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The conjugate gradient method of minimization with an adjoint equation is used successfully to solve the inverse problem in estimating the temperature-dependent thermal conductivity of the homogeneous as well as nonhomogeneous solid material. It is assumed that no prior information is available on the functional form of the unknown thermal conductivity in the present study, thus, it is classified as the function estimation in inverse calculation. The accuracy of the inverse analysis is examined by using simulated exact and inexact measurements obtained within the medium. Results show that an excellent estimation on the thermal conductivity can be obtained with any arbitrary initial guesses by using just boundary measurements (i.e., internal measurements are unnecessary) within 1 s CPU time in a VAX-9420 computer. The advantages of applying this algorithm in inverse analysis can greatly simplify the experimental setup, diminish the sensitivity to the measurement errors, and reduce the CPU time in inverse calculation, while the reliable predictions can still be achieved.

Nomenclature

J = functional defined by Eq. (2)
J' = gradient of functional defined by Eq. (11)
$k(T)$ or $k(x, t)$ = unknown thermal conductivity
P = direction of descent defined by Eq. (3b)
$T(x, t)$ = estimated dimensionless temperature
$Y(x, t)$ = measured temperature
β = search step size
γ = conjugate coefficient
$\Delta T(x, t)$ = sensitivity function defined by Eq. (4)
$\delta(\bullet)$ = Dirac delta function
ε = convergence criteria
$\lambda(x, t)$ = Lagrange multiplier defined by Eq. (9)
ω = random number

Subscript

r = reference parameters

Superscripts

n = iteration index
$\wedge$ = estimated values
$-$ = dimensional parameters

I. Introduction

THE present work addresses the development of an efficient method (i.e., conjugate gradient method) of analysis for estimating the temperature-dependent thermal conductivity of a homogeneous medium using multiple spatial and temporal temperature measurements in transient heat conduction experiments with no prior information on the functional form of the unknown quantities.

Numerous engineering and mathematical researchers have considered problems equivalent to estimating constant or spatially-dependent thermal conductivity, such as in Refs. 1 and 2, which are the special cases of a general class of mathematical problems called "distributed parameter system." Recently, Huang and Ozisik[3,4] used direct integration and Levenberg–Marquardt methods to estimate thermal conductivity and heat capacity simultaneously; Beck and Al-Araji[5] determined the constant thermal conductivity, heat capacity, and contact conductance at one time; Tervola[6] used the Davidon–Fletcher–Powell method to determine temperature-dependent thermal conductivity. All the above references belong to parameter estimations, i.e., the functional form for the unknown quantities should be assigned before the inverse calculations. However, when the thermal conductivity of a nonhomogeneous or composite material is to be estimated, parameter estimation is difficult to achieve, thus, function estimation with the conjugate gradient method should be used in this inverse heat conduction problem to estimate temperature-dependent thermal conductivity $k(T)$.

Alifanov et al.[7,8] were among the earlier users of the conjugate gradient method. More recently, the method has been used for solving the inverse problems of determining the wall heat flux in laminar flow through a parallel plate duct,[9] interface conductance between mold and casting during solidification,[10] interface conductance between periodically contacting surface,[11] wall heat fluxes of a hollow cylinder,[12] and heat fluxes inside the cylinder of an internal combustion engine.[13]

The conjugate gradient method derives basis from the perturbation principles[7] and transforms the inverse problem to the solution of three problems, namely, the direct problem, the sensitivity problem, and the adjoint problem, which will be discussed in detail in this article. In this work, the minimization process with the conjugate gradient method using internal temperature measurements is first examined, then the investigations of using just boundary measurements with simulated exact and inexact temperature are performed and found to be applied successfully in estimating the unknown $k(T)$.

II. Direct Problem

To illustrate the methodology for developing expressions for use in determining unknown $k(T)$ in a homogeneous medium, we consider the following transient inverse heat conduction problem. A slab of thickness $\bar{L}$ is initially at temperature $\bar{T}(\bar{x}, 0) = \bar{T}_0$. For time $\bar{t} > 0$, the boundary surface at $\bar{x} = 0$ is subjected to a prescribed constant heat flux $\bar{q}_1$, while at boundary $\bar{x} = \bar{L}$, a constant heat flux $\bar{q}_2$ is taken away from the slab by cooling. Figure 1a shows the geometry and the

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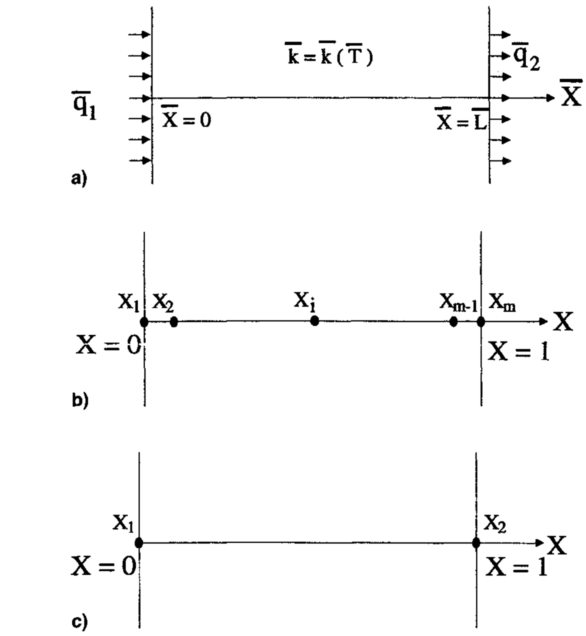

Fig. 1 a) One-dimensional physical problem, b) thermocouple arrangement for *m* points measurements, and c) thermocouple arrangement for 2 points measurements.

coordinates for the one-dimensional physical problem considered here. The mathematical formulation of this transient heat conduction problem in dimensionless form is given by

$$\frac{\partial}{\partial x}\left[k(T)\frac{\partial T(x, t)}{\partial x}\right] = \frac{\partial T(x, t)}{\partial t} \quad \text{in} \quad 0 < x < 1 \tag{1a}$$

$$-k(T)\frac{\partial T(x, t)}{\partial x} = q_1 \quad \text{at} \quad x = 0 \tag{1b}$$

$$-k(T)\frac{\partial T(x, t)}{\partial x} = q_2 \quad \text{at} \quad x = 1 \tag{1c}$$

$$T(x, t) = T_0 \quad \text{for} \quad t = 0 \tag{1d}$$

Where the following dimensionless quantities were defined:

$$x = \bar{x}/\bar{L} \qquad T = \bar{T}/\bar{T}_r \qquad k = \bar{k}/\bar{k}_r$$

$$q = (\bar{L}/\bar{k}_r\bar{T}_r)\bar{q} \qquad t = (\bar{k}_r/\bar{\rho}\bar{C}_p\bar{L}^2)\bar{t}$$

$\bar{\rho}\bar{C}_p$ is the heat capacity per unit volume and $\bar{T}_r$ and $\bar{k}_r$ refer to the nonzero reference temperature and thermal conductivity, respectively. We assume $\bar{T}_0 = \bar{T}_r$, i.e., $T_0 = 1$ in the direct problem (1). The above quantities are assumed known, whereas $k(T)$ is the unknown temperature-dependent thermal conductivity that is yet to be determined.

When generating simulated temperature measurements $T(x, t)$, i.e., given $k(T)$ to calculate temperature $T(x, t)$, the direct problem (1) is nonlinear since thermal conductivities are functions of temperature, therefore, an iterative technique is needed for solving the problem with a finite difference method. At this stage, $k(T)$ cannot be replaced by $k(x, t)$, since $k(T)$ is unknown before the direct problem calculations. However, when the temperatures $T(x, t)$ are converged by iterative technique under some specified initial and boundary conditions, the values of k at any time and position (x, t), should be fixed because temperatures $T(x, t)$ are known and fixed at any (x, t).

Now, in the inverse calculations considered here, the measurement temperatures $T(x, t)$ are assumed known either from numerical simulations or from real experiments. Once $T(x, t)$ results are obtained, there exists some unknown, but fixed, exact thermal conductivity that their values (number), $k(x, t)$, at any specific time and position (x, t), must satisfy the Fourier equation to result in this known temperature distribution $T(x, t)$, therefore, $k(T) \equiv k(x, t)$ can be used in the inverse calculations, even if the problem is nonlinear, i.e., thermal conductivity is a function of temperature.

The direct problem considered here is concerned with the determination of the medium temperature when the thermal conductivity [in the forms of $k(T)$ or $k(x, t)$], other thermal properties, and the boundary conditions at $x = 0$ and $x = 1$ are known.

III. Inverse Problem

For the inverse problem, the thermal conductivity $k(x, t)$ is regarded as being unknown, but everything else in Eq. (1) is known. In addition, temperature readings taken at some appropriate locations are considered available.

Referring to Fig. 1b, we assumed that m sensors are used to record the temperature information to identify $k(x, t)$ in inverse calculations. Let the temperature reading taken within these sensors over the time period t_f be denoted by $Y_i(x_i, t) \equiv Y_i(t)$, $i = 1$ to m, where $i = 1$ and m are always corresponding to $x = 0$ and 1 (i.e., boundary measurements), respectively. We note that the measured temperature $Y_i(t)$ contains measurement errors. Then the inverse problem can be stated as follows: by utilizing the previously mentioned measured temperature data $Y_i(t)$, estimate the unknown $k(x, t)$ over t_f.

Since all the measured temperatures are used to compute the entire unknown function for one period of time variation and no prior information is available on the functional form of $k(x, t)$, therefore, the method used here may be classified as the function estimation in the whole domain[14] for the determination of the nonlinear thermal conductivity $k(T)$.

The solution of the present inverse problem is to be obtained in such a way that the following functional is minimized:

$$J[k(T)] \equiv J[k(x, t)] = \int_{t=0}^{t_f} \sum_{i=1}^{m} [T_i(x_i, t) - Y_i(x_i, t)]^2 \, \mathrm{d}t \tag{2}$$

here, T_i are the estimated temperatures in the slab at the measurement locations $x = x_i$. These quantities are determined from the solution of the direct problem given previously by using an estimated $\hat{k}(x, t)$ for the exact $k(x, t)$. Here, the superscript " ^ " denotes the estimated quantities.

IV. Conjugate Gradient Method for Minimization

The following iterative process based on the conjugate gradient method[7] is now used for the estimation of $k(x, t)$ by minimizing the functional $J[k(x, t)]$

$$\hat{k}^{n+1}(x, t) = \hat{k}^n(x, t) - \beta^n P^n(x, t) \quad \text{for} \quad n = 0, 1, 2, \ldots \tag{3a}$$

where β^n is the search step size in going from iteration n to iteration $n + 1$, and $P^n(x, t)$ is the direction of descent (i.e., search direction) given by

$$P^n(x, t) = J'^n(x, t) + \gamma^n P^{n-1}(x, t) \tag{3b}$$

which is a conjugation of the gradient direction $J'^n(x, t)$ at iteration n and the direction of descent $P^{n-1}(x, t)$ at iteration $n - 1$. The conjugate coefficient is determined from

$$\gamma^n = \int_{x=0}^{1}\int_{t=0}^{t_f} (J'^n)^2 \, \mathrm{d}t \, \mathrm{d}x \Bigg/ \int_{x=0}^{1}\int_{t=0}^{t_f} (J'^{n-1})^2 \, \mathrm{d}t \, \mathrm{d}x$$

$$\text{with} \quad \gamma^0 = 0 \tag{3c}$$

We note that when $\gamma^n = 0$ for any n, in Eq. (3b), the direction of descent $P^n(x, t)$ becomes the gradient direction, i.e., the "steepest descent" method is obtained. The convergence of the previous iterative procedure in minimizing the functional J is guaranteed in Ref. 15.

To perform the iterations according to Eqs. (3), we need to compute the step size β^n and the gradient of the functional $J'^n(x, t)$. In order to develop expressions for the determination of these two quantities, a "sensitivity problem" and an "adjoint problem" are constructed as described next.

V. Sensitivity Problem and Search Step Size

The sensitivity problem is obtained from the original direct problem defined by Eq. (1) in the following manner: It is assumed that when $k(x, t)$ undergoes a variation $\Delta k(x, t)$, $T(x, t)$ is perturbed by $T + \Delta T$. Then replacing in the direct problem k by $k + \Delta k$ and T by $T + \Delta T$, subtracting from the resulting expressions the direct problem, and neglecting the second-order terms, the following sensitivity problems for the sensitivity function ΔT are obtained:

$$\frac{\partial}{\partial x}\left[k(x, t)\frac{\partial \Delta T(x, t)}{\partial x}\right] + \frac{\partial}{\partial x}\left[\Delta k(x, t)\frac{\partial T(x, t)}{\partial x}\right] = \frac{\partial \Delta T(x, t)}{\partial t} \quad \text{in} \quad 0 < X < 1 \tag{4a}$$

$$-k(x, t)\frac{\partial \Delta T(x, t)}{\partial x} = \Delta k(x, t)\frac{\partial T(x, t)}{\partial x} \quad \text{at} \quad x = 0 \tag{4b}$$

$$-k(x, t)\frac{\partial \Delta T(x, t)}{\partial x} = \Delta k(x, t)\frac{\partial T(x, t)}{\partial x} \quad \text{at} \quad x = 1 \tag{4c}$$

$$\Delta T(x, t) = 0 \quad \text{for} \quad t = 0 \tag{4d}$$

The functional $J(\hat{k}^{n+1})$ for iteration $n + 1$ is obtained by rewriting Eq. (2) as

$$J(\hat{k}^{n+1}) = \int_{t=0}^{t_f} \sum_{i=1}^{m} [T_i(\hat{k}^n - \beta^n P^n) - Y_i]^2 \, dt \tag{5a}$$

where we replaced $\hat{k}^{n+1}$ by the expression given by Eq. (3a). If temperature $T_i(\hat{k}^n - \beta^n P^n)$ is linearized by a Taylor expansion, Eq. (5a) takes the form

$$J(\hat{k}^{n+1}) = \int_{t=0}^{t_f} \sum_{i=1}^{m} [T_i(\hat{k}^n) - \beta^n \Delta T_i(p^n) - Y_i]^2 \, dt \tag{5b}$$

where $T_i(\hat{k}^n)$ is the solution of the direct problem by using estimate $\hat{k}^n(x, t)$ for exact $k(x, t)$ at $x = x_i$. The sensitivity function $\Delta T_i(P^n)$ is taken as the solutions of problem (4) at the measured positions $x = x_i$ by letting $\Delta k = P^n$.[16] The search step size β^n is determined by minimizing the functional given by Eq. (5b) with respect to β^n. The following expression results:

$$\beta^n = \int_{t=0}^{t_f} \sum_{i=1}^{m} (T_i - Y_i)\Delta T_i \, dt \Bigg/ \int_{t=0}^{t_f} \sum_{i=1}^{m} (\Delta T_i)^2 \, dt \tag{6}$$

VI. Adjoint Problem and Gradient Equation

To obtain the adjoint problem, Eq. (1a) is multiplied by the Lagrange multiplier (or adjoint function) $\lambda(x, t)$ and the resulting expression is integrated over the time and correspondent space domains. Then the result is added to the right-hand side (RHS) of Eq. (2) to yield the following expression for the functional $J[k(x, t)]$:

$$J[k(x, t)] = \int_{t=0}^{t_f} \sum_{i=1}^{m} [T_i - Y_i]^2 \, dt + \int_{x=0}^{1}\int_{t=0}^{t_f} \lambda \left\{\frac{\partial}{\partial x}\left[k(T)\frac{\partial T(x, t)}{\partial x}\right] - \frac{\partial T(x, t)}{\partial t}\right\} dt \, dx \tag{7}$$

The variation ΔJ is obtained by perturbing k by Δk and T by ΔT in Eq. (7), subtracting from the resulting expression the original Eq. (7) and neglecting the second-order terms. We thus find

$$\begin{aligned}\Delta J = &\int_{t=0}^{t_f} 2(T_1 - Y_1)\Delta T_1 \, dt + \int_{x=0}^{1}\int_{t=0}^{t_f} \sum_{i=2}^{m-1} 2(T - Y)\Delta T \delta(x - x_i) \, dt \, dx + \int_{t=0}^{t_f} 2(T_m - Y_m)\Delta T_m \, dt \\ &+ \int_{x=0}^{1}\int_{t=0}^{t_f} \lambda \left\{\frac{\partial}{\partial x}\left[k(x, t)\frac{\partial \Delta T(x, t)}{\partial x}\right] + \frac{\partial}{\partial x}\left[\Delta k(x, t)\frac{\partial T(x, t)}{\partial x}\right] - \frac{\partial \Delta T(x, t)}{\partial t}\right\} dt \, dx\end{aligned} \tag{8}$$

where $\delta(x - x_i)$ is the Dirac delta function and x_i, $i = 2$ to $m - 1$, refers to the internal measured positions; they were shown to be numerically unnecessary in Sec. IX. In Eq. (8), the second double integral term is integrated by parts; the initial and boundary conditions of the sensitivity problem given by Eqs. (4b–4d) are utilized and then ΔJ is allowed to go to zero. The vanishing of the integrands containing ΔT leads to the following adjoint problem for the determination of $\lambda(x, t)$:

$$\frac{\partial}{\partial x}\left[k(x, t)\frac{\partial \lambda(x, t)}{\partial x}\right] + \sum_{i=2}^{m-1} 2(T - Y)\delta(x - x_i) + \frac{\partial \lambda(x, t)}{\partial t} = 0 \quad \text{in} \quad 0 < X < 1 \tag{9a}$$

$$-k(x, t)\frac{\partial \lambda(x, t)}{\partial x} = 2(T_1 - Y_1) \quad \text{at} \quad x = 0 \tag{9b}$$

$$k(x, t)\frac{\partial \lambda(x, t)}{\partial x} = 2(T_m - Y_m) \quad \text{at} \quad x = 1 \tag{9c}$$

$$\lambda(x, t) = 0 \quad \text{for} \quad t = t_f \tag{9d}$$

The adjoint problem is different from the standard initial value problems in that the final time condition at time $t = t_f$ is specified instead of the customary initial condition. However, this problem can be transformed to an initial value problem by the transformation of the time variables as $\tau = t_f - t$. Then the standard techniques can be used to solve the adjoint problem.

Finally, the following integral term is left:

$$\Delta J = \int_0^L \int_{t=0}^{t_f} -\left[\frac{\partial \lambda(x, t)}{\partial x}\frac{\partial T(x, t)}{\partial x}\right] \Delta k(x, t) \, dt \, dx \tag{10a}$$

Before obtaining the expression of gradient equation, we first introduce a notation of gradient of functional (the first Frechet derivative[17]). For $k = k(x, t)$, $x \in [0, 1]$ and $t \in [0, t_f]$, is the function considered as an element of functional space $k(x, t) \in L_2$-norm. If the functional increment can be presented as

$$\Delta J = \int_0^L \int_{t=0}^{t_f} J'(x, t)\Delta k(x, t) \, dt \, dx \tag{10b}$$

Then function $J'(x, t)$ is called a gradient of functional. A comparison of Eqs. (10a) and (10b) leads to the following expression for the gradient $J'(x, t)$ of the functional $J[k(x, t)]$:

$$J'(x, t) = -\frac{\partial \lambda(x, t)}{\partial x}\frac{\partial T(x, t)}{\partial x} \tag{11}$$

We note that $J'(x, t_f)$ is always equal to zero since $\lambda(x, t_f) = 0.0$, therefore, if the final time values of $k(x, t_f)$ can't be predicted before the inverse calculation, the estimated values of $k(x, t)$ will deviate from exact values near the final time condition.[7] This is the case in the present study. However, if we let $\lambda(x, t_f) = \lambda(x, t_f - \Delta t)$, where Δt denotes the time increment for use in finite difference calculations, the singularity at $t = t_f$ can be avoided in the present study and reliable inverse solutions can be obtained.

VII. Stopping Criterion

If the problem contains no measurement errors, the traditional check condition is specified as

$$J[\hat{k}^{n+1}(x, t)] < \varepsilon \tag{12}$$

where ε is a small specified number. However, the observed temperature data may contain measurement errors. Therefore, we do not expect the functional Eq. (2) to be equal to zero at the final iteration step. Following the experience of the authors,[9-13] we use the discrepancy principle as the stopping criterion, i.e., we assume that the temperature residuals may be approximated by

$$T_i - Y_i \approx \sigma \tag{13}$$

where σ is the stand deviation of the measurements, which is assumed to be a constant. This assumption was also made by Tikhonov[18] in order to find the optimal regularization parameter. Substituting Eq. (13) into Eq. (2), the following expression is obtained for ε:

$$\varepsilon = m\sigma^2 t_f \tag{14}$$

Then, the stopping criterion is given by Eq. (12) with ε determined from Eq. (14).

VIII. Computational Procedure

The computational procedure for the solution of this inverse problem may be summarized as follows:

Suppose $\hat{k}^n(x, t)$ is available at iteration n.

Step 1: solve the direct problem given by Eq. (1) for $T(x, t)$. Since $\hat{k}(x, t)$ rather than $k(T)$ is used, therefore, the problem becomes linear.

Step 2: examine the stopping criterion given by Eq. (12) with ε given by Eq. (14). Continue if not satisfied.

Step 3: solve the adjoint problem given by Eq. (9) for $\lambda(x, t)$.

Step 4: compute the gradient of the functional J' from Eq. (11).

Step 5: compute the conjugate coefficient γ^n and direction of descent P^n from Eqs. (3c) and (3b), respectively.

Step 6: set $\Delta k(x, t) = P^n(x, t)$, and solve the sensitivity problem given by Eq. (4) for $\Delta T(x, t)$.

Step 7: compute the search step size β^n from Eq. (6).

Step 8: compute the new estimation for $\hat{k}^{n+1}(x, t)$ from Eq. (3a) and return to step 1.

IX. Results and Discussion

To illustrate the accuracy of the conjugate gradient method in predicting $k(T)$ with inverse analysis from the knowledge of transient temperature recordings, we consider two specific examples where the thermal conductivity is in a very complicated functional form, such as the combination of sinusoidal and exponential function in the first example and the combination of second-order polynomial and exponential function in the second example, with temperature as the dependent variable.

The objective of this article is to show the validity of the present approach in estimating $k(T)$ accurately with no prior information on the functional form of the unknown quantities, which is the so-called function estimation. Moreover, the internal temperature measurements will be shown unnecessary from the numerical experiments for both examples.

In order to compare the results for situations involving random measurement errors, we assume normally distributed uncorrelated errors with zero mean and constant standard deviation. The simulated inexact measurement data Y can be expressed as

$$Y = Y_{\text{exact}} + \omega\sigma \tag{15}$$

where Y_{exact} is the solution of the direct problem with an exact $k(T)$; σ is the standard deviation of the measurements; and ω is a random variable that is generated by subroutine DRNNOR of the IMSL,[19] and will be within -2.576 to 2.576 for a 99% confidence bounds. One should note that when generating simulated measurement temperature Y, exact $k(T)$ is used in the direct problem, and thus, the problem is nonlinear and the iterative technique is needed for its solutions. However, in the inverse calculation, the thermal conductivity exists in the form of $k(x, t)$, so that the problem becomes linear and the estimated temperature can be calculated directly.

One of the advantages of using the conjugate gradient method is that the initial guesses of the unknown quantities can be chosen arbitrarily. In all the test cases considered here, the initial guesses of $\hat{k}(x, t)$ used to begin the iteration are taken as $\hat{k}(x, t)_{\text{initial}} = 10^{-8}$. The space and time increments are taken as $\Delta x = 0.1$ and $\Delta t = 0.1$, respectively, in the finite difference calculations; the boundary heat fluxes are taken as $q_1 = 20$ and $q_2 = 14$; the total measurement time is chosen as $t_f = 6$, and the measurement time step Dt is taken the same as Δt, therefore, a total of 660 discrete numbers of $k(x, t)$ are to be estimated in the inverse calculations.

We now present two numerical experiments in determining $k(T)$ by the inverse analysis.

A. Numerical Test Case 1

The thermal conductivity $k(T)$ is assumed to vary with temperature in the form

$$k(T) = K_0 + K_1 \times \exp(T/K_2) + K_3 \times \sin(T/K_4) \tag{16}$$

where the constants K_0, K_1, K_2, K_3, and K_4 are taken as 1, 4.5, 80, 2.5, and 5, respectively. The exact function of $k(T)$ in terms of $k(x, t)$ within the total space and time domain is sketched in Fig. 2.

The inverse analysis is first performed by using 11 thermocouple measurements (referring to Fig. 1b with $m = 11$) with thermocouple spacing Dx equal to finite difference spacing Δx, i.e., $Dx = 0.1$. When assuming exact measurements,

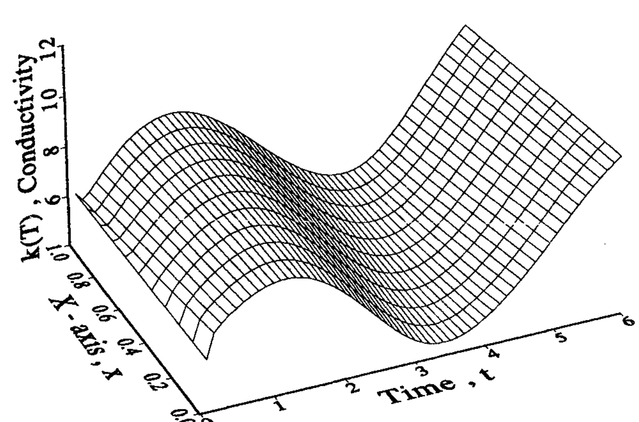

Fig. 2 Exact function of $k(T)$ in case 1. $k(T) = K_0 + K_1 \times \exp(T/K_2) + K_3 \times \sin(T/K_4)$.

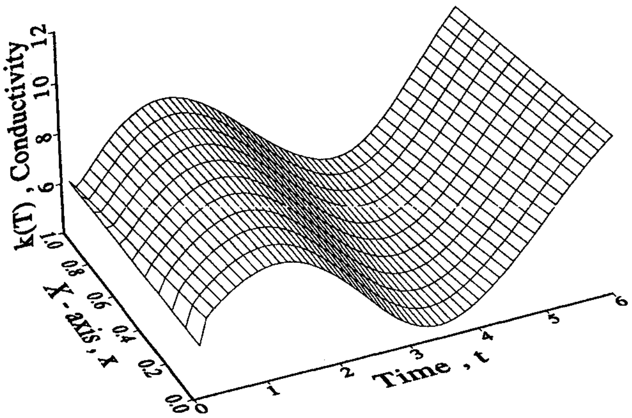

Fig. 3 Estimated function of $k(x, t)$ in case 1 by 11 sensors with σ = 0.0.

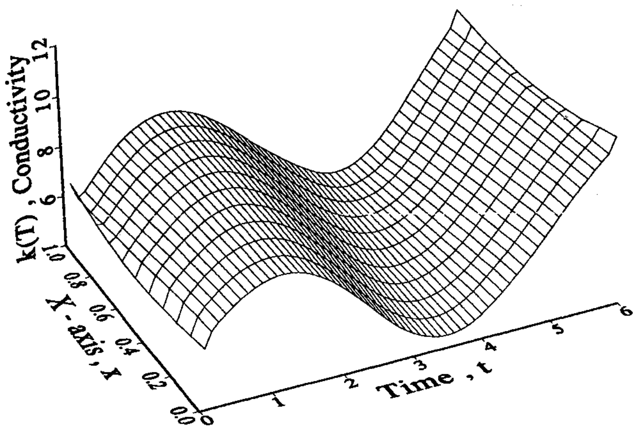

Fig. 4 Estimated function of $k(x, t)$ in case 1 by 2 sensors with σ = 0.0.

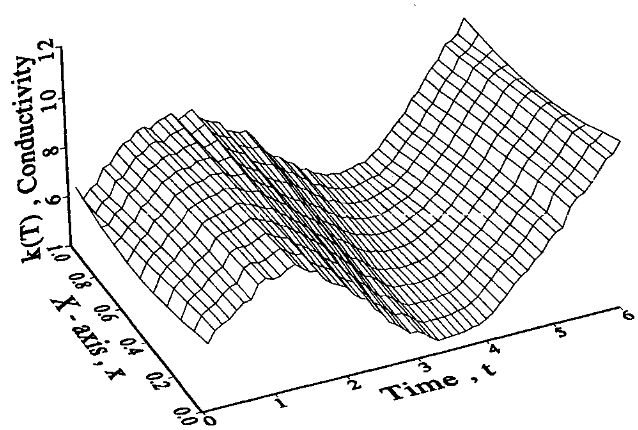

Fig. 5 Estimated function of $k(x, t)$ in case 1 by 2 sensors with σ = 0.01.

σ = 0.0, the estimated function of $k(x, t)$ is shown in Fig. 3. The value of functional J obtained in such a case can be decreased to a very small number as the number of iterations are increased. The comparisons between Figs. 2 and 3 show that the inverse analysis with conjugate gradient method in estimating $k(x, t)$ are now accomplished.

However, this test case seems unrealistic, since too many internal sensors were used in the numerical experiment. Now the question arises, can the number of sensors be reduced with the present approach? To answer this, the numerical experiment is then proceeded to the second case, i.e., using just boundary measurements (referring to Fig. 1c with m = 2) and σ = 0.0. We note that the internal heat sources appearing in the adjoint problem should be deleted in this case.

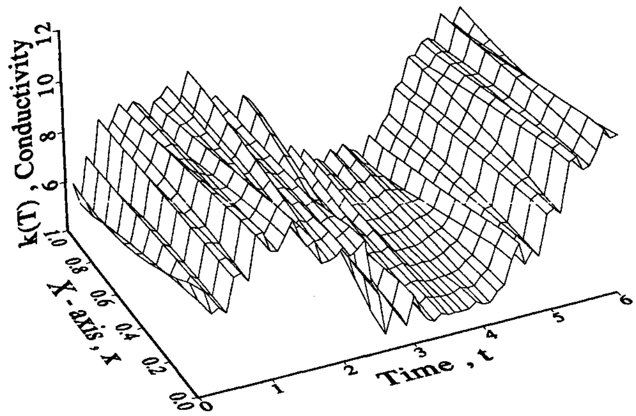

Fig. 6 Estimated function of $k(x, t)$ in case 1 by 2 sensors with σ = 0.07.

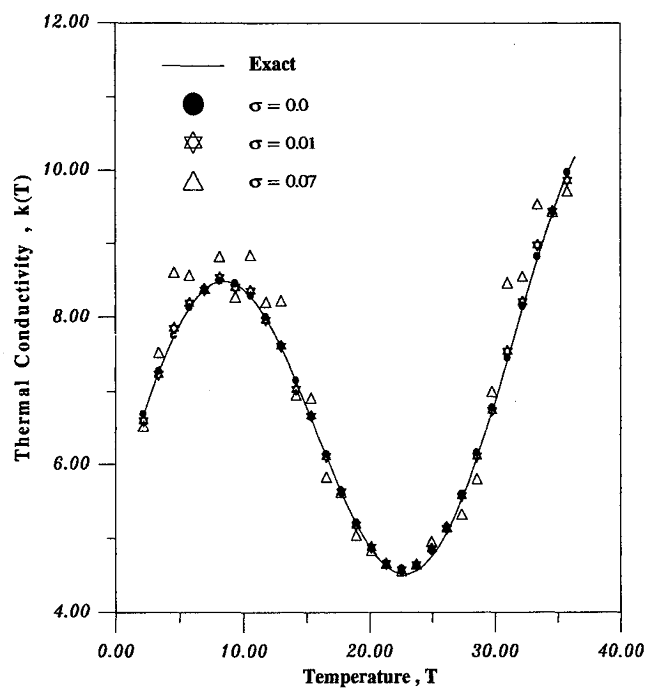

Fig. 7 Exact and estimated values of $k(T)$ at x = 0.5 in case 1.

The inverse solutions in predicting $k(x, t)$ under such an assumption is shown in Fig. 4, which is also in an excellent agreement with the exact function of $k(x, t)$ as shown in Fig. 2. From the comparisons of numerical data we learned that the inverse solutions in predicting $k(x, t)$ with 11 sensors are better than that with 2 sensors, especially near the final time region, however, the latter case is already good enough to be accepted as the inverse solutions.

Next, when the dimensionless measured temperatures with errors σ = 0.01 and σ = 0.07 are obtained according to Eq. (15), the inverse solutions using these inexact measurements as the simulated temperature measurements are shown in Figs. 5 and 6, respectively. Moreover, in order to show $k(T)$ more explicitly as a function of temperature T and the temperature range of measurements, the thermal conductivity $k(T)$ vs T at x = 0.5 with measurement errors σ = 0, 0.01, and 0.07 is presented in Fig. 7.

The average relative errors between exact and estimated values are 1.350 and 5.564% for σ = 0.01 and 0.07, respectively, where the average relative error is defined as

$$\left(\sum_{i=1}^{m}\sum_{j=1}^{n}\left|\frac{k(x_i, t_j) - \hat{k}(x_i, t_j)}{k(x_i, t_j)}\right|\right) \div (n \times m) \times 100\% \qquad (17)$$

Table 1 Convergent parameters for 2 point measurements in case 1

Case 1: $k(T) = K_0 + K_1 \times \exp(T/K_2) + K_3 \times \sin(T/K_4)$				
Measurement error, σ	Stopping criterion	Number of iterations	VAX-9420 CPU time, s	Average relative error, %
0.00	1.00 E-004	28	0.49	0.662
0.01	1.20 E-003	21	0.36	1.350
0.07	5.88 E-002	17	0.31	5.564

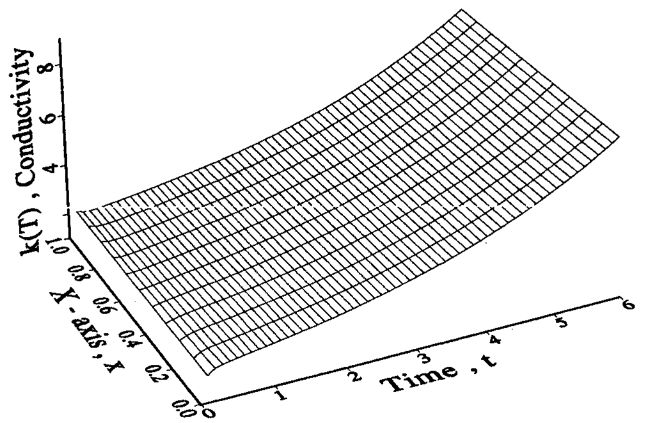

Fig. 8 Exact function of $k(x, t)$ in case 2. $k(T) = K_0 + K_1 \times T + K_2 \times T^2 + \exp(T/K_3)$.

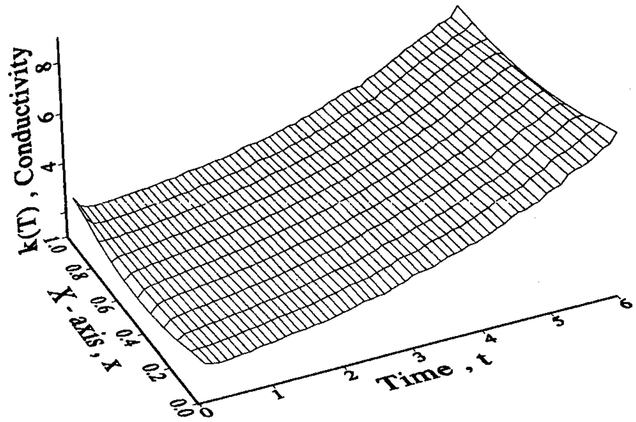

Fig. 9 Estimated function of $k(x, t)$ in case 2 by 2 sensors with $\sigma = 0.01$.

and m and n represent the total discrete number of position and time increments, respectively, while k and $\hat{k}$ denote the exact and estimated values of thermal conductivity.

For the case when $\sigma = 0.07$, the dimensionless measured temperature errors will be within -0.18 to 0.18 for a 99% confidence bounds, which implies that a total of about 0.36 dimensionless temperature error is allowed. According to Fig. 7, the dimensionless temperature at $x = 0.5$ will range from 1 to 35, thus, the average relative measurement error is about 2%. By using this 2% measurement error, one could estimate the thermal conductivity with an average relative error of about 6%. This proves that the measurement errors did not amplify the errors of estimated thermal conductivity, and therefore, the present technique provides a confident estimation.

To illustrate the fast convergence of the present approach, Table 1 shows the number of iterations and CPU time in a VAX-9420 computer for 2-point measurements. Indeed, the necessary CPU time is always within 1 s and the number of iterations are in the order of one-tenth in estimating 660 unknown discrete numbers of $k(x, t)$ simultaneously by using arbitrary initial guesses, which shows that the speed of convergence is very fast.

B. Numerical Test Case 2

In the second test case, $k(T)$ is taken as

$$k(T) = K_0 + K_1 \times T + K_2 \times T^2 + \exp(T/K_3) \quad (18)$$

where the constants K_0, K_1, K_2, and K_3 are taken as 1, 0.01, 0.00001, and 20, respectively, and the exact function of $k(x, t)$ is shown in Fig. 8.

The estimation of $k(x, t)$ by using 11 and 2 sensors with exact measurements $\sigma = 0.0$ also shows a very good agreement with the exact values of $k(x, t)$, except for the values near $t = 0$ and $t = t_f$, when only 2 sensors were used. Therefore, the three-dimensional plots for both cases are omitted since they will be very similar to Fig. 8.

Next, when the measurement errors with $\sigma = 0.01$ and 0.1 are considered, the estimations for $k(T)$ are sketched in Figs.

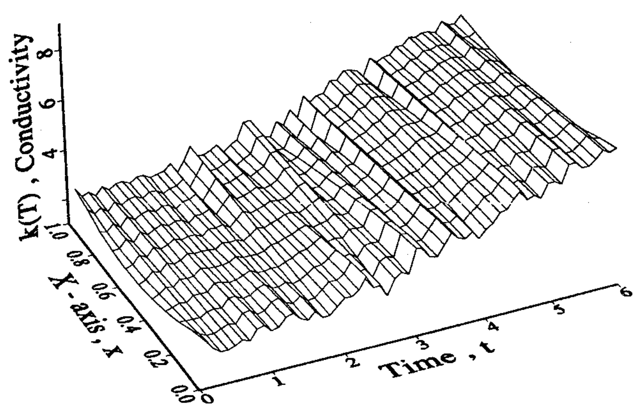

Fig. 10 Estimated function of $k(x, t)$ in case 2 by 2 sensors with $\sigma = 0.1$.

9 and 10, respectively, Fig. 11 indicates the relation between $k(T)$ and T at $x = 0.5$. The average relative errors between exact and estimated values are 1.199 and 5.030% for $\sigma = 0.01$ and 0.1, respectively. For the case when $\sigma = 0.1$, the dimensionless measured temperature errors will be within -0.2576 to 0.2576 for a 99% confidence bounds, this means a total of about 0.5 dimensionless temperature error is allowed. From Fig. 11, the dimensionless temperature at $x = 0.5$ will range from 1 to 35, thus the average relative measurement error is about 3%. Again, this result shows that the measurement errors did not amplify the error of estimated thermal conductivity.

Table 2 indicates the CPU time and number of iterations for 2 point measurements, which also shows a fast convergence in using the conjugate gradient method for the inverse calculations. Finally, if one is interested in computing the confidence bounds of the present study, the technique used in Refs. 3 and 4 is recommended.

From the previous numerical test cases 1 and 2, we concluded that the conjugate gradient method can be applied successfully in the function estimation for predicting the temperature-dependent thermal conductivity with very fast speed of convergence.

Table 2 Convergent parameters for 2 point measurements in case 2

Case 2: $k(T) = K_0 + K_1 \times T + K_2 \times T^2 + \exp(T/K_3)$				
Measurement error, σ	Stopping criterion	Number of iterations	VAX-9420 CPU time, s	Average relative error, %
0.00	1.00 E-004	37	0.62	0.975
0.01	1.20 E-003	29	0.48	1.199
0.10	1.20 E-001	17	0.31	5.030

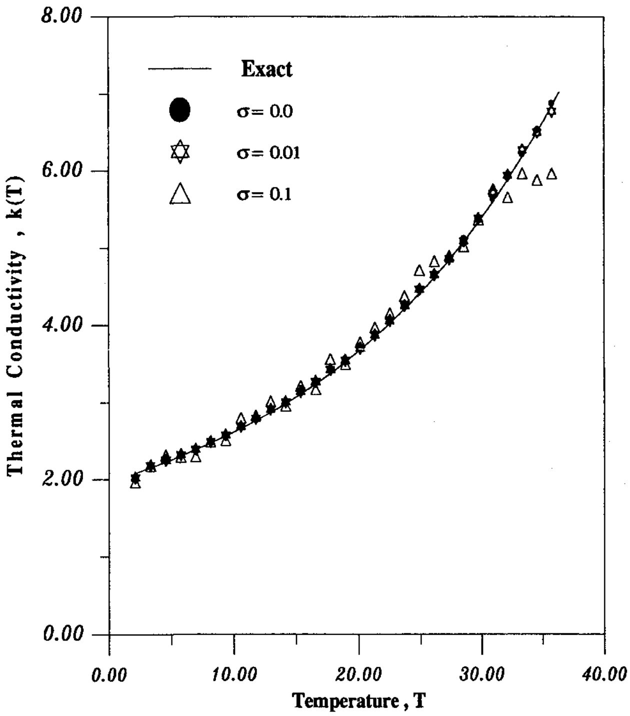

Fig. 11 Exact and estimated values of $k(T)$ at $x = 0.5$ in case 2.

X. Conclusions

The conjugate gradient method with adjoint equation was successfully applied for the solution of the inverse problem to determine the temperature-dependent thermal conductivity without the necessity of using interior sensors. Several test cases involving different functional forms of $k(T)$ and measurement errors were considered. The results show that the conjugate gradient method does not require a priori information for the functional form of the unknown quantities and it needs a very short CPU time in the VAX-9420 to perform the inverse calculations.

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